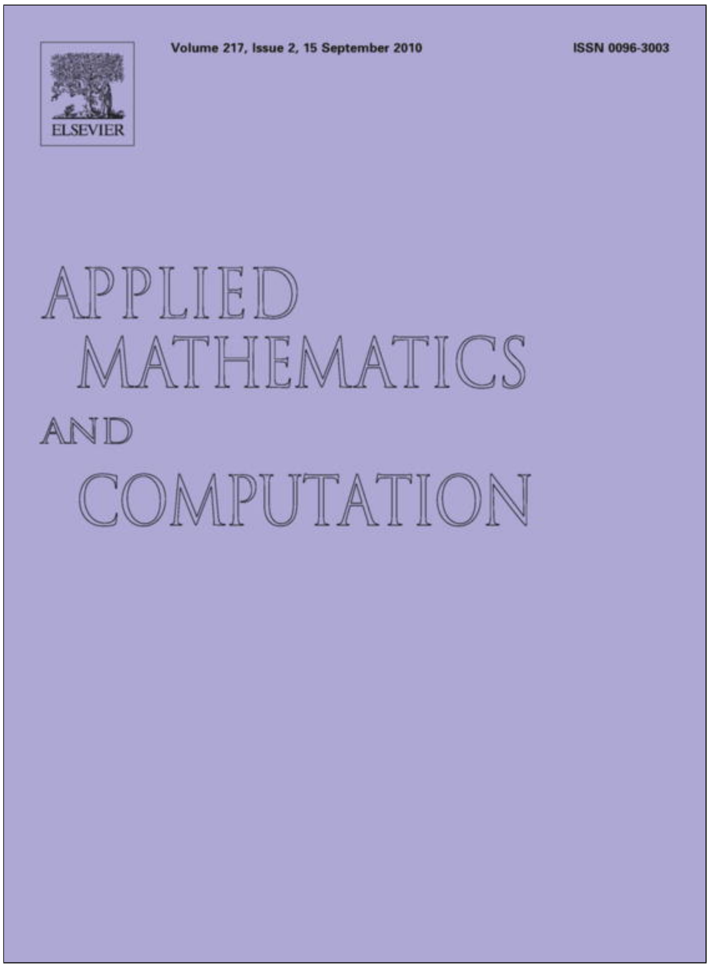
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Approximate solutions to the nonlinear vibrations of multiwalled carbon nanotubes using Adomian decomposition method

N.H. Sweilam [a,*], M.M. Khader [b]

[a] Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[b] Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

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ABSTRACT

This paper applies the Adomian decomposition method (ADM) to the search for the approximate solutions to the problem of the nonlinear vibrations of multiwalled carbon nanotubes embedded in an elastic medium. A multiple-beam model is utilized in which the governing equations of each layer are coupled with those of its adjacent ones via the van der Waals inter layer forces. The amplitude–frequency curves for large-amplitude vibrations of single-walled, double-walled and triple-walled carbon nanotubes are obtained. The influence of changes in material constants of the surrounding elastic medium and the effect of changes in nanotube geometrical parameters on the vibration characteristics are studied by comparing the results with those from the open literature. This method needs less work in comparison with the traditional methods and decreases considerable volume of calculation, and it's powerful mathematical tool for solving wide class of nonlinear differential equations. Special attention is given to prove the convergence of the method. Some examples are given to illustrate the determination approximate solutions of the proposed problem.

1. Introduction

In the past few years much attention devoted to simulate real-life problems which can be described by nonlinear differential equations using reliable and more efficient methods. The ADM (see [1–4,6,7,9–11] and the references cited therein) is one of these methods which has received much concern, it has the merits of simplicity and easy execution. Unlike the traditional numerical methods, ADM does not need small parameter, discretization, linearization, transformations or perturbation. The solution procedure by ADM is very simple, and only few terms lead to high accuracy solutions which are valid for the solution domain. The nonlinear coupled systems of partial differential equations often appear in the study of circled fuel reactor, high-temperature hydrodynamics and thermo-elasticity problems, see [12–15]. From the analytical point of view, a lot of work has been done for such systems.

With the rapid development of nanotechnology, there appears an ever-increasing interest of scientists and researchers in this field of science. Nanomaterials, because of their exceptional mechanical, physical and chemical properties have been the main topic of research in many scientific publications. Nowadays, they are used as the substantial parts of nanoelectronics, nanodevices, and nanocomposites. One of these materials attracted great attention due to its high mechanical strength is carbon nanotube (CNT). CNTs were discovered by Iijima [8] in 1991. In spite of being too small and having light weight, they have very large Young's modulus in axial direction (nearly 1TPa). Undoubtedly, CNTs have the eligibility to be the new and

* Corresponding author.
E-mail addresses: n_sweilam@yahoo.com (N.H. Sweilam), mohamedmbd@yahoo.com (M.M. Khader).

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most popular nanomaterial of this early part of the 21st century. Since the vibration of CNTs are of considerable importance in a number of nanomechanical devices such as oscillators, charge detectors, field emission devices and sensors, Many researches have been so far devoted to the problem of the vibration of these Nanomaterials ([17–19]). However, most of the investigations conducted on the vibration of multiwalled carbon nanotubes (MWNTs) have been restricted to the linear regime and fewer works were done on the nonlinear vibration of these materials. Recently Fu [5] studied the nonlinear vibrations of embedded nanotubes using the incremental harmonic balanced method (IHBM). In that work, single-walled nanotubes (SWNTs) and double-walled nanotubes (DWNTs) considered for the study. The goal of this paper is to use the merits of simplicity of ADM to search the approximate solutions for triple-walled nanotubes (TWNTs).

The paper is organized as follows: In Section 2 we introduce the fundamentals of Adomian decomposition method. In Section 3 we implement ADM to obtain the approximate solutions of multiwalled carbon nanotubes, and the convergence of the ADM is presented. The last section gives a discussion of our results.

2. Fundamentals of Adomian decomposition method

In this section, a brief outline of ADM is explained. For this, we consider a general nonlinear differential equation in the following form:

$$Lu + Ru + N(u) = g, \tag{1}$$

where L is the highest order derivative which is assumed to be easily invertible, R linear differential operator of less order than L, $N(u)$ presents the nonlinear term and g is the source term. Applying the inverse operator L^{-1} to the both sides of (1) and using the given conditions we obtain:

$$u = \varphi(x) + L^{-1}[g] - L^{-1}[Ru + N(u)], \tag{2}$$

where the function $\varphi(x)$ presents the solution of the homogenous differential equation $Lu = 0$, using the given conditions. The ADM defines the solution u by the series in the following form:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{3}$$

and the nonlinear operator $N(u)$ presents by an infinite series of the so-called Adomian's polynomials:

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

where $u_n(x)$, $n \geqslant 0$ are the components of $u(x)$ that will be elegantly determined and A_n are called Adomian's polynomials and defined by:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n \geqslant 0. \tag{5}$$

From the above considerations, the decomposition method defines the components $u_n(x)$ for $n \geqslant 0$, by the following recursive relationships:

$$u_0(x) = \varphi(x) + L^{-1}[g], \quad u_{n+1}(x) = -L^{-1}[Ru_n + A_n], \quad n \geqslant 0. \tag{6}$$

This will enable us to determine the components u_n recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solution $u(x)$ such that:

$$\lim_{n\to\infty} U_n(x) = u(x), \quad \text{where } U_n(x) = \sum_{i=0}^{n-1} u_i(x), \quad n \geqslant 0. \tag{7}$$

For more details about ADM and its convergence see [11,14,16].

3. Solution procedure using ADM

In this section, we apply ADM to obtain the approximate solution to the problem of the nonlinear vibrations of CNTs.

Case 1: Nonlinear vibration of a SWNT.

Consider a SWNT of length l, Young's modulus E, density ρ, cross-sectional area A, and cross-sectional inertia moment I, embedded in an elastic medium with material constant k. The nonlinear vibration equation for this CNT is in the following form [5]:

$$\frac{d^2W}{dt^2} + \left(\frac{\pi^4 EI}{\rho A l^4} + \frac{k}{\rho A}\right) W + \frac{\pi^4 E}{4\rho l^4} W^3 = 0, \tag{8}$$

under the transformations: $r = \sqrt{\frac{I}{A}}\, x = \frac{W}{r},\ w_l = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}},\ w_k = \sqrt{\frac{k}{\rho A}}\ \tau = \omega t$, the above equation can be transformed to the following dimensionless nonlinear vibration equation:

$$\omega^2 \frac{d^2x}{d\tau^2} + w_b^2 x + \alpha w_l^2 x^3 = 0, \tag{9}$$

in which $\alpha = 0.25$ and $w_b = \sqrt{w_l^2 + w_k^2}$, is the linear, free vibration frequency. With the initial conditions:

$$x(0) = X, \quad \dot{x}(0) = 0. \tag{10}$$

First, we rewrite (9) in an operator form as follows:

$$L(x) + N(x) = 0, \tag{11}$$

where the notations; $L = \frac{d^2}{dt^2}$ symbolize the linear differential operator, N is nonlinear operator and defined by:

$$N(x) = w_b^2 x + \alpha w_l^2 x^3. \tag{12}$$

By using the inverse operator, we can write (11) in the following form:

$$x(t) = x(0) - L^{-1}[N(x)], \tag{13}$$

where the inverse operator is defined by: $L^{-1}(.) = \int_0^t \int_0^t (.) dt\, dt$.

The ADM suggests that the solution can be decomposed by an infinite series of components:

$$x(t) = \sum_{n=0}^{\infty} x_n(t) \tag{14}$$

and the nonlinear term (12) decomposed by the infinite series (4), where $x_n(t)$, $n \geqslant 0$ are the components of that will be elegantly determined and A_n, $n \geqslant 0$ are called Adomian's polynomials defined by (5).

Now, by using the above considerations, the decomposition method defines the components $x_n(t)$, $n \geqslant 0$ by the following recursive relationships:

$$x_0(t) = x(0), \quad x_{n+1}(t) = -L^{-1}[A_n], \quad n \geqslant 0. \tag{15}$$

This will enable us to determine the components recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparison purpose, we construct the solution $x(t)$ such that:

$$\lim_{n\to\infty} \Theta_n(t) = x(t), \quad \text{where } \Theta_n(t) \simeq \sum_{i=0}^{n-1} x_i(t), \quad n \geqslant 0. \tag{16}$$

In order to seek the periodic solution of Eq. (9) assume the initial approximation to be the linear solution of Eq. (9) as:

$$x_0(t) = X\cos(\psi w_b t). \tag{17}$$

This initial approximation is a trial function and it is used to obtain more accurate approximate solution of Eq. (9). Here ψ, is the ratio of the nonlinear frequency, ω, to the linear frequency, w_b. Substituting the initial approximation into Eq. (9) results in the following residual:

$$R_0(\xi) = (-X\psi^2 w_b^2 + w_b^2 X + 0.75\alpha w_l^2 X^3)\cos(\psi w_b \xi) + 0.25\alpha w_l^2 X^3 \cos(3\psi w_b \xi). \tag{18}$$

In order to ensure that no secular terms appear in the next iteration, the coefficient of $\cos(\psi w_b \xi)$ must vanish. Therefore:

$$\psi = \sqrt{1 + \frac{3}{4}\alpha\left(\frac{w_l}{w_b}\right)^2 X^2}. \tag{19}$$

To find the first component of the solution of (9) using ADM, we calculate the first Adomian's polynomial using Eq. (5) as follows:

$$A_0 = w_b^2 x_0 + \alpha w_l^2 x_0^3,$$

then, by using the given initial condition (10), we can derive the first component of the solution in the following form:

$$x_1(t) = \frac{-X}{(w_b\psi)^2}\left[w_b^2 + 0.195X^2 w_l^2 - (w_b^2 + 0.1875X^2 w_l^2)\cos(\psi w_b t) - 0.00695X^2 w_l^2 \cos(3\psi w_b t)\right],$$

with ψ defined as in Eq. (19). The amplitude–frequency response curves for a SWNT for different spring k constants are shown in Fig. 1. The material and geometric parameters taken here are $E = 1.1TPa$, $\rho = 1300\ \text{kg/m}^3$, $l = 45$ nm, the outer diameter $d_1 = 3$ nm and the inner diameter $d_0 = 2.32$ nm. In Fig. 1, ψ is the ratio of nonlinear frequency to linear frequency as discussed earlier and X is the maximum vibration amplitude. It can be seen that as the spring constant k increases, the nonlinear

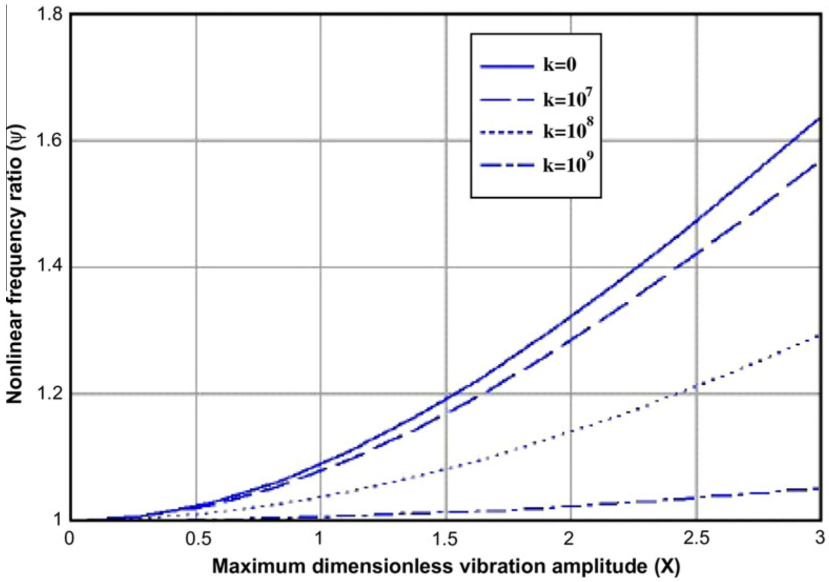

Fig. 1. Effect of spring constant k on nonlinear amplitude–frequency response curves of SWNT.

frequencies tend to approach the linear ones especially when exceeds the value 10^7 n/m^2. It should be noted that Fig. 1 is exactly the same as the figure obtained via incremental harmonic balance method (IHBM) [5].

3.1. Convergence analysis of ADM

In this section, we study the convergence analysis of ADM to the solution when applied to the model problem (8). Let us define the Hilbert space $H = L^2(\alpha,\beta)$ as a set of all applications:

$$W : (\alpha,\beta) \to \Re \quad \text{with} \quad \int_{(\alpha,\beta)} |W|^2(s)\, ds < \infty.$$

Let us consider $L(W) = \frac{d^2W}{dt^2}$, then we can rewrite (8) in the following operator form:

$$\frac{d^2W}{dt^2} = -\lambda_1 W - \lambda_2 W^3, \tag{20}$$

where $\lambda_1 = \frac{\pi^4 EI}{\rho A l^4} + \frac{k}{\rho A}$, $\lambda_2 = \frac{\pi^4 E}{4\rho l^4}$.

Theorem 1. *The ADM applied to the nonlinear (8) converges towards a particular solution if the following two hypotheses are satisfied:*

$$(H1) : (L(W) - L(U), W - U) \geqslant m\|W - U\|^2, \quad m > 0, \quad \forall W, U \in H,$$
$$(H2) : \text{there exist} C(K) > 0, K > 0, \text{such that} \quad \forall W, U \in H \text{ with } \|W\| \leqslant K, \|U\| \leqslant K,$$

we have $(L(W) - L(U), P) \leqslant C(K)\|W - U\|\|P\| \quad \forall P \in H.$

Proof. To verify $(H1)$ for the operator $L(W)$, we have:

$$L(W) - L(U) = -\lambda_1(W - U) - \lambda_2(W^3 - U^3).$$

Then we claim:

$$(L(W) - L(U), W - U) = -\lambda_1(W - U, W - U) - \lambda_2(W^3 - U^3, W - U). \tag{21}$$

Now, we have:

$$(W - U, W - U) \leqslant \|W - U\|\|W - U\| = \|W - U\|^2, \tag{22}$$

where $W < \eta < U$ and $\|W\| < K, \|U\| < K$. Therefore:

$$(W^3 - U^3, W - U) \leqslant \|W^3 - U^3\|\|W - U\| = 3\eta^2\|W - U\|^2 = 3K^2\|W - U\|^2, \tag{23}$$

substituting from (22) and (23) into (21), we get:

$$(L(W) - L(U), W - U) \geqslant (\lambda_1 + 3\lambda_2 K^2)\|W - U\|^2 = m\|W - U\|^2,$$

where $m = \lambda_1 + 3\lambda_2 K^2 = \frac{\pi^4 EI}{\rho A l^4} + \frac{k}{\rho A} + 3\frac{\pi^4 E}{4\rho l^4}K^2$. Hence, we verified ($H1$). To verify ($H2$) for the operator $L(W)$, we have:

$$(L(W) - L(U), P) = -\lambda_1(W - U, P) - \lambda_2(W^3 - U^3, P), \tag{24}$$

therefore,

$$(L(W) - L(U), P) \leqslant (\lambda_1 + 3\lambda_2 K^2)\|W - U\|\|P\| = C(K)\|W - U\|\|P\|,$$

where $C(K) = \lambda_1 + 3\lambda_2 K^2$. Hence, we verified ($H2$) and the end of the proof. □

Case 2: Nonlinear vibration of a DWNT.

The nonlinear vibration governing equation for a DWNT is in the following form [5]:

$$\frac{d^2W_1}{dt^2} + \left(\frac{\pi^4 EI_1}{\rho A_1 l^4} + \frac{c_1}{\rho A_1}\right)W_1 + \frac{\pi^4 E}{4\rho l^4}W_1^3 - \frac{c_1}{\rho A_1}W_2 = 0, \tag{25}$$

$$\frac{d^2W_2}{dt^2} + \left(\frac{\pi^4 EI_2}{\rho A_2 l^4} + \frac{c_1}{\rho A_2} + \frac{k}{\rho A_2}\right)W_2 + \frac{\pi^4 E}{4\rho l^4}W_2^3 - \frac{c_1}{\rho A_2}W_1 = 0, \tag{26}$$

where c_1 is the coefficient of the van der Waals force between the i-th tube and the i-1th tube. By substituting the following dimensionless parameters:

$$r = \sqrt{\frac{I_1}{A_1}},\ x = \frac{W_1}{r},\ y = \frac{W_2}{r},\ \omega_l = \frac{\pi^2}{l^2}\sqrt{\frac{EI_1}{\rho A_1}},\ \omega_k = \sqrt{\frac{k}{\rho A_1}},\ \omega_c = \sqrt{\frac{c}{\rho A_1}},\ \tau = wt, \beta = \frac{A_1}{A_2},\ \gamma = \frac{I_1}{I_2},\ \alpha = 0.25.$$

Eqs. (25) and (26) can be transformed to the following dimensionless nonlinear system:

$$\left(\frac{\omega}{\omega_l}\right)^2 \frac{d^2x}{d\tau^2} + B_1 x + \alpha x^3 - B_2 y = 0, \tag{27}$$

$$\left(\frac{\omega}{\omega_l}\right)^2 \frac{d^2y}{d\tau^2} + B_3 y + \alpha y^3 - B_4 x = 0, \tag{28}$$

with B_1 to B_4 defined as:

$$B_1 = 1 + \left(\frac{\omega_c}{\omega_l}\right)^2, \quad B_2 = \left(\frac{\omega_c}{\omega_l}\right)^2, \quad B_3 = \beta\left(\frac{1}{\gamma} + \left(\frac{\omega_c}{\omega_l}\right)^2 + \left(\frac{\omega_k}{\omega_l}\right)^2\right), \quad B_4 = \beta\left(\frac{\omega_c}{\omega_l}\right)^2.$$

With the initial conditions:

$$x(0) = X_1, \quad y(0) = X_2, \quad \dot{x}(0) = \dot{y}(0) = 0. \tag{29}$$

First, we write (27) and (28) in an operator form as follows:

$$L(x) + N(x, y) = 0, \tag{30}$$

$$L(y) + M(x, y) = 0, \tag{31}$$

where the nonlinear operators are defined by:

$$N(x, y) = \omega_l^2 B_1 x + \alpha\omega_l^2 x^3 - \omega_l^2 B_2 y, \quad M(x, y) = \omega_l^2 B_3 y + \alpha\omega_l^2 y^3 - \omega_l^2 B_4 x. \tag{32}$$

By using the inverse operator, we can write (30) and (31) in the following form:

$$x(t) = x(0) - L^{-1}[N(x, y)], \quad y(t) = y(0) - L^{-1}[M(x, y)]. \tag{33}$$

The ADM suggests that the solutions $x(t)$ and $y(t)$ can be decomposed by an infinite series of components:

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{34}$$

and the nonlinear terms which defined in (32) decomposed by the infinite series:

$$N(x, y) = \sum_{n=0}^{\infty} A_n, \quad M(x, y) = \sum_{n=0}^{\infty} B_n, \tag{35}$$

where $x_n(t)$ and $y_n(t)$, $n \geqslant 0$ are the components of $x(t)$ and $y(t)$ respectively, that will be elegantly determined and A_n, B_n, $n \geqslant 0$ are called Adomian's polynomials and defined by (5).

Now, by using the above considerations, the decomposition method defines the components $x_n(t)$ and $y_n(t)$, $n \geqslant 0$ by the following recursive relationships:

$$x_0(t) = x(0), \quad x_{n+1}(t) = -L^{-1}[A_n], \quad n \geqslant 0, \tag{36}$$
$$y_0(t) = y(0), \quad y_{n+1}(t) = -L^{-1}[B_n], \quad n \geqslant 0. \tag{37}$$

This will enable us to determine the components $x_n(t)$ and $y_n(t)$ recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparison purpose, we construct the solutions $x(t)$ and $y(t)$ such that:

$$\lim_{n\to\infty} \Theta_n(t) = x(t), \text{ and } \lim_{n\to\infty} \Psi_n(t) = y(t), \tag{38}$$

where $\Theta_n(t) = \sum_{i=0}^{n-1} x_i(t)$, and $\Psi_n(t) = \sum_{i=0}^{n-1} y_i(t)$, $\quad n \geqslant 0$.

In order to seek the periodic solutions of Eqs. (27) and (28) assume the initial approximations to be the linear solution of Eqs. (27) and (28) as:

$$x_0(t) = X_1 \cos(\psi w_b t), \quad y_0(t) = X_2 \cos(\psi w_b t). \tag{39}$$

These initial approximations are trial functions and it used to obtain more accurate approximate solutions of Eqs. (27) and (28). Substituting the initial approximations into Eqs. (27) and (28) result in the following residuals:

$$R_{10}(\xi) = (-X_1\psi^2 w_b^2 + w_l^2 B_1 X_1 + 0.75\alpha w_l^2 X_1^3 - B_2 w_l^2 X_2)\cos(\psi w_b \xi) + 0.25\alpha w_l^2 X_1^3 \cos(3\psi w_b \xi),$$
$$R_{20}(\xi) = (-X_2\psi^2 w_b^2 + w_l^2 B_3 X_2 + 0.75\alpha w_l^2 X_2^3 - B_4 w_l^2 X_1)\cos(\psi w_b \xi) + 0.25\alpha w_l^2 X_2^3 \cos(3\psi w_b \xi).$$

Here in ψ, the ratio of the nonlinear frequency ω to the linear frequency ω_b, is the unknown constant. Following the same approach as above and also eliminating the coefficient of $\cos(\psi w_{b\ t})$ in the above system due to avoiding the secular terms, results in the following nonlinear system which can be easily solved using a simple mathematical algorithm such as Newton–Raphson technique.

$$-\left(\frac{\psi}{\omega_l}\right)^2 X_1 w_b^2 + B_1 X_1 + \frac{3}{4}\alpha X_1^3 - B_2 X_2 = 0, \tag{40}$$
$$-\left(\frac{\psi}{\omega_l}\right)^2 X_2 w_b^2 + B_3 X_2 + \frac{3}{4}\alpha X_2^3 - B_4 X_1 = 0. \tag{41}$$

To calculate the linear vibration frequencies for DWNT, we shall first substitute $x = X_1\cos(\psi w_{b\ t})$ and $y = X_2\cos(\psi w_{b\ t})$ into Eqs. (27) and (28) without considering the nonlinear terms in Eqs. (27) and (28), so that:

$$\begin{pmatrix} \omega_l^2 + \omega_c^2 - \omega^2 & -\omega_c^2 \\ -\beta\omega_c^2 & \beta\left(\frac{\omega_l^2}{\gamma} + \omega_c^2 + \omega_k^2\right) - \omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{42}$$

Then by setting the determinant of the matrix in Eq. (42) equal to zero, the frequency characteristic equation will be obtained. The fundamental linear vibration frequency of DWNT is the lowest root of the resulting equation. Fig. 2 shows the variation of the nonlinear amplitude–frequency response curves of DWNT against the maximum vibration amplitude for different spring constants k. The material and geometric parameters used to obtain this figure are, E = 1.1 TPa, ρ = 1300 kg/m^3,

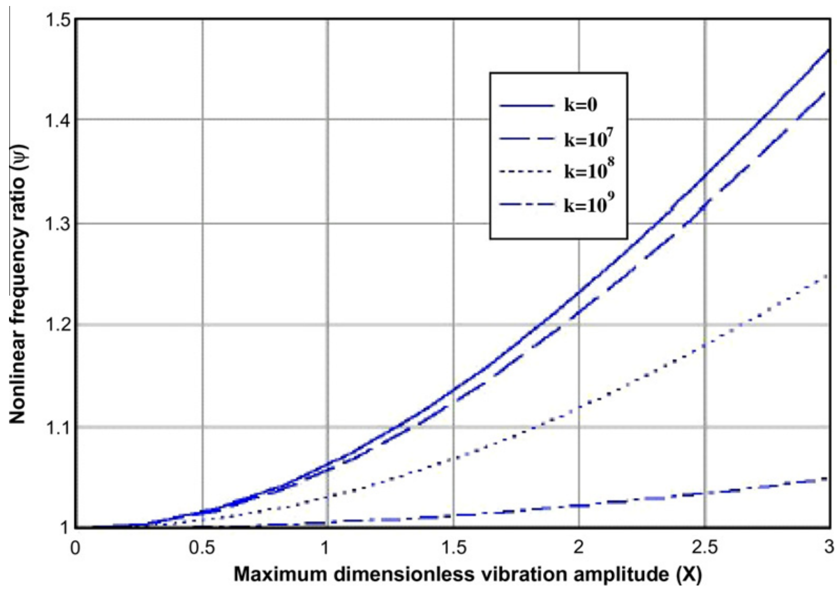

Fig. 2. Effect of spring constant k on nonlinear amplitude–frequency response curves of DWNT.

$c = 0.3 \times 10^{12}$ N/m^2, l = 45 nm, d_0 = 1.64 nm, d_1 = 2.32 nm and d_2 = 3 nm. It can be seen that the effect of spring constant on nonlinear vibration of DWNT is similar to that emerged in the case of SWNT (Fig. 1) and this figure is exactly the same figure as that obtained via (IHBM) [5].

To find the first component of the solutions of Eqs. (27) and (28) using ADM, we calculate the first Adomian's polynomials A_0 and B_0 using Eq. (5) as follows:

$$A_0 = w_l^2 B_1 x_0 + \alpha w_l^2 x_0^3 - w_l^2 B_2 y_0, \quad B_0 = w_l^2 B_3 y_0 + \alpha w_l^2 y_0^3 - w_l^2 B_4 x_0,$$

then, by using the given initial condition (29), we can derive the first component of the solution in the following form:

$$x_1(t) = -\left(\frac{\omega_l}{w_b\psi}\right)^2 \left[0.195X_1^3 + B_1X_1 - B_2X_2 + \left(-0.1874X_1^3 - B_1X_1 - B_2X_2\right)\cos(\psi w_b t) - 0.00695X_1^3\cos(3\psi w_b t)\right],$$

$$y_1(t) = -\left(\frac{\omega_l}{w_b\psi}\right)^2 \left[0.195X_2^3 + B_3X_2 - B_4X_1 + \left(-0.1874X_2^3 - B_3X_2 - B_4X_1\right)\cos(\psi w_b t) - 0.00695X_2^3\cos(3\psi w_b t)\right].$$

Case 3: Nonlinear vibration of a TWNT.

The nonlinear vibration governing equations for TWNTs are in the following form:

$$\frac{d^2W_1}{dt^2} + \left(\frac{\pi^4 EI_1}{\rho A_1 l^4} + \frac{c_1}{\rho A_1}\right)W_1 + \frac{\pi^4 E}{4\rho l^4}W_1^3 - \frac{c_1}{\rho A_1}W_2 = 0, \tag{43}$$

$$\frac{d^2W_2}{dt^2} + \left(\frac{\pi^4 EI_2}{\rho A_2 l^4} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right)W_2 + \frac{\pi^4 E}{4\rho l^4}W_2^3 - \frac{c_1}{\rho A_2}W_1 - \frac{c_2}{\rho A_2}W_3 = 0, \tag{44}$$

$$\frac{d^2W_3}{dt^2} + \left(\frac{\pi^4 EI_3}{\rho A_3 l^4} + \frac{c_1}{\rho A_3} + \frac{c_2}{\rho A_3} + \frac{k}{\rho A_3}\right)W_3 + \frac{\pi^4 E}{4\rho l^4}W_3^3 - \frac{c_2}{\rho A_3}W_2 = 0. \tag{45}$$

In a similar manner, introducing the following dimensionless parameters:

$$r = \sqrt{\frac{I_1}{A_1}},\ x = \frac{W_1}{r},\ y = \frac{W_2}{r},\ z = \frac{W_3}{r},\ \omega_l = \frac{\pi^2}{l^2}\sqrt{\frac{EI_1}{\rho A_1}},\ \omega_k = \sqrt{\frac{k}{\rho A_1}},\ \omega_c = \sqrt{\frac{c}{\rho A_1}},\ \tau = wt, \beta = \frac{A_1}{A_2},\ \gamma = \frac{I_1}{I_2},\ \eta = \frac{A_1}{A_3},$$
$$\zeta = \frac{I_1}{I_3},\ \alpha = 0.25,$$

to the Eqs. (43)–(45) leads to the dimensionless nonlinear vibration equations as:

$$\omega^2\frac{d^2x}{d\tau^2} + \omega_l^2 B_1 x + \alpha\omega_l^2 x^3 - \omega_l^2 B_2 y = 0, \tag{46}$$

$$\omega^2\frac{d^2y}{d\tau^2} + \omega_l^2 B_3 y + \alpha\omega_l^2 y^3 - \omega_l^2\beta B_2 x - \omega_l^2\beta B_2 z = 0, \tag{47}$$

$$\omega^2\frac{d^2z}{d\tau^2} + \omega_l^2 B_4 z + \alpha\omega_l^2 z^3 - \omega_l^2\eta B_2 y = 0, \tag{48}$$

with B_1 to B_4 defined as:

$$B_1 = 1 + \left(\frac{\omega_c}{\omega_l}\right)^2, \quad B_2 = \left(\frac{\omega_c}{\omega_l}\right)^2, \quad B_3 = \beta\left(\frac{1}{\gamma} + 2\left(\frac{\omega_c}{\omega_l}\right)^2\right), \quad B_4 = \eta\left(\frac{1}{\zeta} + 2\left(\frac{\omega_c}{\omega_l}\right)^2 + \left(\frac{\omega_k}{\omega_l}\right)^2\right).$$

With the initial conditions:

$$x(0) = X_1, \quad y(0) = X_2, \quad z(0) = X_3, \quad \dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0. \tag{49}$$

First, we write (46)–(48) in an operator form as follows:

$$L(x) + N(x,y,z) = 0, \quad L(y) + M(x,y,z) = 0, \quad L(z) + V(x,y,z) = 0. \tag{50}$$

where the nonlinear operators are defined by:

$$\begin{aligned} N(x,y,z) &= \omega_l^2 B_1 x + \alpha\omega_l^2 x^3 - \omega_l^2 B_2 y, \\ M(x,y,z) &= \omega_l^2 B_3 y + \alpha\omega_l^2 y^3 - \omega_l^2 B_2\beta x - \omega_l^2\beta B_2 z, \\ V(x,y,z) &= \omega_l^2 B_4 z + \alpha\omega_l^2 z^3 - \omega_l^2\eta B_2 y. \end{aligned} \tag{51}$$

By using the inverse operator, we can write (50) in the following form:

$$x(t) = x(0) - L^{-1}[N(x,y,z)], \quad y(t) = y(0) - L^{-1}[M(x,y,z)], \quad z(t) = z(0) - L^{-1}[V(x,y,z)]. \tag{52}$$

The ADM suggests that the solutions can be decomposed by an infinite series of components:

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \quad y(t) = \sum_{n=0}^{\infty} y_n(t), \quad z(t) = \sum_{n=0}^{\infty} z_n(t) \tag{53}$$

and the nonlinear terms which defined in (51) decomposed by the infinite series:

$$N(x,y,z) = \sum_{n=0}^{\infty} A_n, \quad M(x,y,z) = \sum_{n=0}^{\infty} B_n, \quad V(x,y,z) = \sum_{n=0}^{\infty} C_n, \tag{54}$$

where $x_n(t)$, $y_n(t)$ and $z_n(t)$, $n \geqslant 0$ are the components of $x(t)$, $y(t)$ and $z(t)$ respectively, that will be elegantly determined and A_n, B_n, C_n, $n \geqslant 0$ are called Adomian's polynomials and defined by (5). Now, by using the above considerations, the decomposition method defines the components $x_n(t)$, $y_n(t)$, and $z_n(t)$, $n \geqslant 0$ by the following recursive relationships:

$$x_0(t) = x(0), \quad x_{n+1}(t) = -L^{-1}[A_n], \quad n \geqslant 0, \tag{55}$$

$$y_0(t) = y(0), \quad y_{n+1}(t) = -L^{-1}[B_n], \quad n \geqslant 0, \tag{56}$$

$$z_0(t) = z(0), \quad z_{n+1}(t) = -L^{-1}[C_n], \quad n \geqslant 0. \tag{57}$$

This will enable us to determine the components $x_n(t)$, $y_n(t)$ and $z_n(t)$ recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solutions $x(t)$ and $y(t)$ such that:

$$\lim_{n\to\infty} \Theta_n(t) = x(t), \quad \lim_{n\to\infty} \Psi_n(t) = y(t) \text{ and } \lim_{n\to\infty} \Omega_n(t) = z(t), \tag{58}$$

where $\Theta_n(t) = \sum_{i=0}^{n-1} x_i(t)$, $\Psi_n(t) = \sum_{i=0}^{n-1} y_i(t)$, and $\Omega_n(t) = \sum_{i=0}^{n-1} z_i(t), n \geqslant 0$.

In order to seek the periodic solutions of Eqs. (46)–(48) assume the initial approximations to be the linear solution of Eqs. (46)–(48) as:

$$x_0(t) = X_1 \cos(\psi w_b t), \quad y_0(t) = X_2 \cos(\psi w_b t), \quad z_0(t) = X_3 \cos(\psi w_b t). \tag{59}$$

These initial approximations are trial functions and it used to obtain more accurate approximate solutions of Eqs. (46)–(48). Substituting the initial approximations into Eqs. (46)–(48) result in the following residuals:

$$R_{10}(\xi) = \left(-X_1\psi^2 w_b^2 + w_l^2 B_1 X_1 + 0.75\alpha w_l^2 X_1^3 - B_2 w_l^2 X_2\right)\cos(\psi w_b \xi) + 0.25\alpha w_l^2 X_1^3 \cos(3\psi w_b \xi),$$

$$R_{20}(\xi) = \left(-X_2\psi^2 w_b^2 + (B_3 X_2 + 0.75\alpha X_2^3 - B_2\beta X_1 - B_2\beta X_3)\right) w_l^2 \cos(\psi w_b \xi) + 0.25\alpha w_l^2 X_2^3 \cos(3\psi w_b \xi),$$

$$R_{30}(\xi) = \left(-X_3\psi^2 w_b^2 + (B_4 X_3 + 0.75\alpha X_3^3 - B_2\eta X_2)\right) w_l^2 \cos(\psi w_b \xi) + 0.25\alpha w_l^2 X_3^3 \cos(3\psi w_b \xi).$$

Here in ψ, the ratio of the nonlinear frequency ω to the linear frequency ω_b, is the unknown constant. Following the same approach as above and also eliminating the coefficient of $\cos(\psi w_{b\ t})$ in the above system due to avoiding the secular terms, results in the following nonlinear system which can be easily solved using a simple mathematical algorithm such as Newton–Raphson technique.

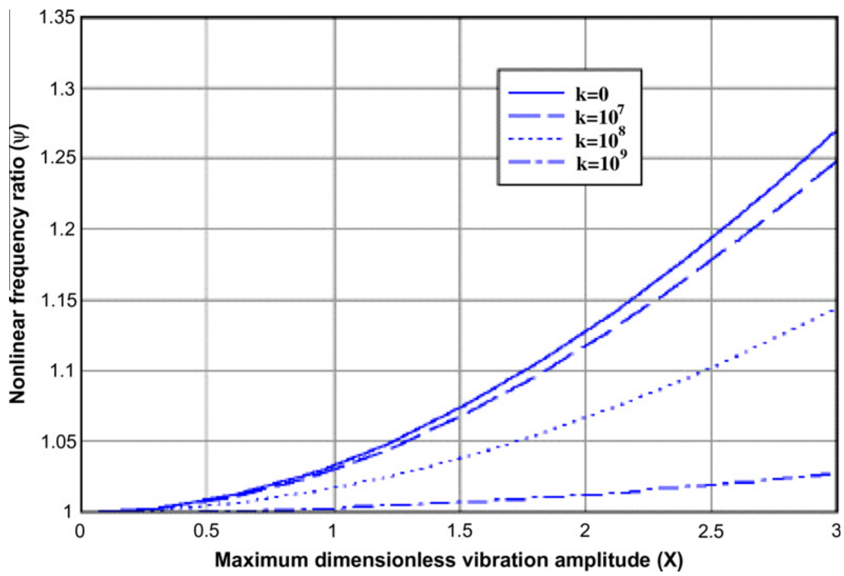

Fig. 3. Effect of spring constant k on nonlinear amplitude–frequency response curves of TWNT.

$$-\left(\frac{\psi}{\omega_l}\right)^2 X_1 w_b^2 + B_1 X_1 + \frac{3}{4}\alpha X_1^3 - B_2 X_2 = 0, \tag{60}$$

$$-\left(\frac{\psi}{\omega_l}\right)^2 X_2 w_b^2 + B_3 X_2 + \frac{3}{4}\alpha X_2^3 - B_2 \beta X_1 - B_2 \beta X_3 = 0, \tag{61}$$

$$-\left(\frac{\psi}{\omega_l}\right)^2 X_3 w_b^2 + B_4 X_3 + \frac{3}{4}\alpha X_3^3 - B_2 \eta X_2 = 0. \tag{62}$$

To calculate the linear vibration frequencies for TWNT, we shall first substitute $x = X_1\cos(\psi\omega_b t)$, $y = X_2\cos(\psi\omega_b t)$, and $z = X_3\cos(\psi\omega_b t)$, into Eqs. (46)–(48) without considering the nonlinear terms in Eqs. (46)–(48), so that:

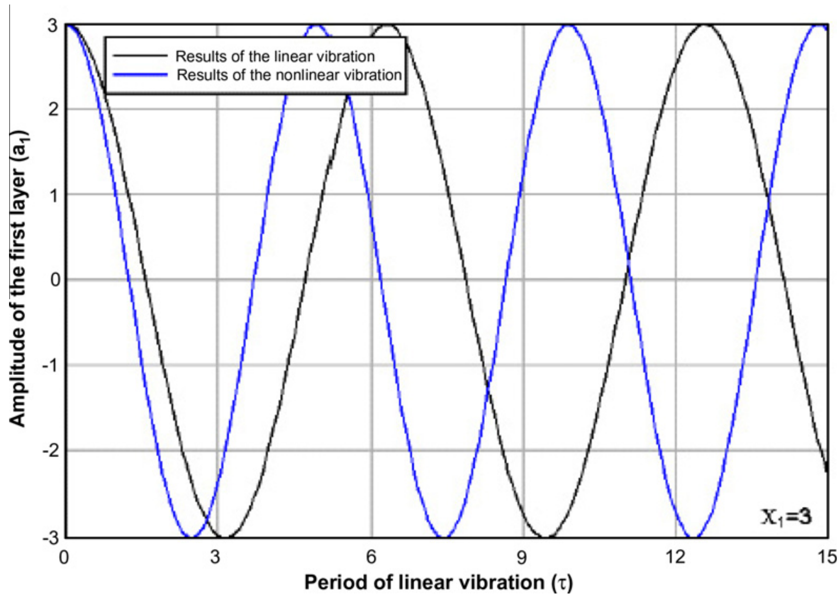

Fig. 4. The nonlinear amplitude of the vibration of the first layer of TWNT.

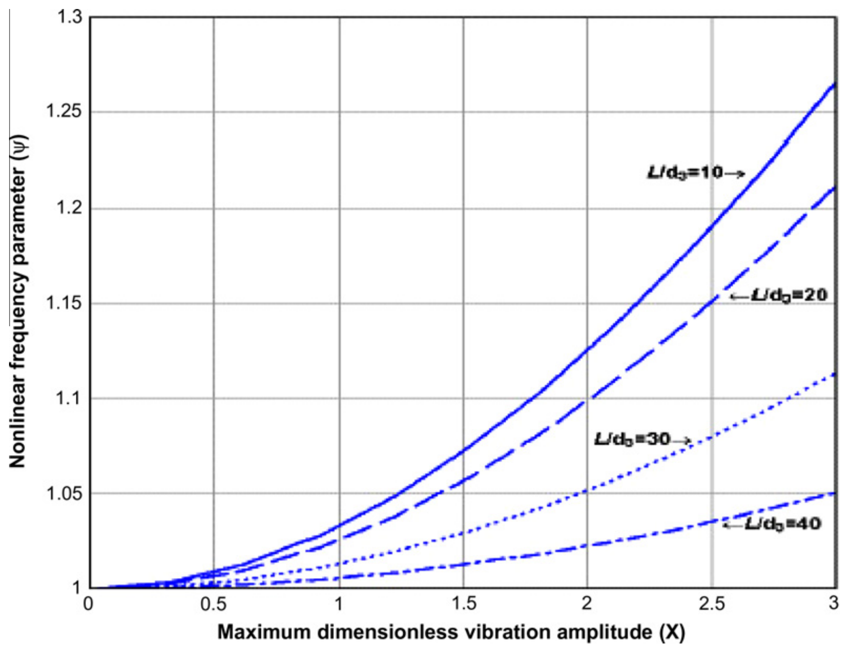

Fig. 5. Effect of aspect ratio L/d_2 on nonlinear amplitude–frequency response curves for TWNT.

Table 1
The linear free vibration frequencies ω_b of SWNT, DWNT and TWNT in Figs. 1–3.

k (N/m^2)	ω_b(TH$_z$)		
	SWNT	DWNT	TWNT
0	0.128	0.116	0.111
10^7	0.138	0.122	0.117
10^8	0.209	0.170	0.156
10^9	0.536	0.410	0.365

$$\begin{pmatrix} \omega_l^2+\omega_c^2-\omega^2 & -\omega_c^2 & 0 \\ -\beta\omega_c^2 & \beta\left(\frac{\omega_l^2}{\gamma}+2\omega_c^2\right)-\omega^2 & -\beta\omega_c^2 \\ 0 & -\eta\omega_c^2 & \eta\left(\frac{\omega_l^2}{\zeta}+\omega_c^2+\omega_k^2\right)-\omega^2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{63}$$

For a nontrivial solution to exist, the determinant of the above matrix must be vanished which leads to the frequency characteristic equation to be solved. The fundamental linear vibration frequency of TWNT is the lowest root of the resulting equation. The variation of the nonlinear amplitude–frequency response curves of TWNT against the maximum vibration amplitude for different spring constants is also illustrated in Fig. 3. The material and geometric parameters used are $c_1 = c_2 = 0.3 \times 10^{12}$ N/m^2, $l = 45$ nm, $d_0 = 0.96$ nm, $d_1 = 1.64$ nm, $d_2 = 2.32$ nm and $d_3 = 3$ nm. Clearly the same behavior as above is indefeasible in the case of TWNT. A comparison between the amplitude of the nonlinear vibration of the first layer of TWNT with its linear vibration amplitude is shown in Fig. 4 for $X_1 = 3$ and $k = 0$ against the linear period of vibration ($\tau = \omega_b t$). Its worthwhile to say that the discrepancy between the linear and nonlinear amplitudes increases with the increment of the maximum amplitude. In Fig. 5, the parameters are $k = 10^7$ N/m^2, $c = 0.3 \times 10^{12}$ N/m^{23} and $d_2 = 3$ nm. It is observed that with the increase of the aspect ratio of the nanotubes, the nonlinear vibration frequencies of MWNTs decrease. Due to convenience in calculating the nonlinear free vibration frequency ω, the linear vibration frequencies ω_b of SWNT, DWNT and TWNT for all cases are listed in Table 1.

To find the first component of the solution of Eqs. (46)–(48) using ADM, we calculate the first Adomian's polynomials A_0, B_0 and C_0 using Eq. (5) as follows:

$$A_0 = w_l^2 B_1 x_0 + \alpha w_l^2 x_0^3 - w_l^2 B_2 y_0,$$
$$B_0 = w_l^2 B_3 y_0 + \alpha w_l^2 y_0^3 - w_l^2 B_2 \beta x_0 - w_l^2 \beta z_0,$$
$$C_0 = w_l^2 B_4 z_0 + \alpha w_l^2 z_0^3 - w_l^2 B_2 \eta y_0.$$

Then, by using the given initial conditions, we can derive the first components of the solutions in the following form:

$$x_1(t) = -\left(\frac{\omega_l}{w_b\psi}\right)^2\left[0.195X_1^3 + B_1X_1 - B_2X_2 + (-0.1874X_1^3 - B_1X_1 - B_2X_2)\cos(\psi w_b t) - 0.00695X_1^3\cos(3\psi w_b t)\right],$$
$$y_1(t) = -\left(\frac{\omega_l}{w_b\psi}\right)^2\left[-(B_3X_2 + 0.1875X_2^3 - B_2(X_1+X_3)(-1+\cos(\psi w_b t)) + 0.00695X_2^3(1-\cos(3\psi w_b t))\right],$$
$$z_1(t) = -\left(\frac{\omega_l}{w_b\psi}\right)^2\left[0.195X_{32}^3 + B_4X_3 - B_2\eta X_2 + \left(-0.1874X_3^3 - B_4X_3 + B_2\eta X_2\right)\cos(\psi w_b t) - 0.00695X_3^3\cos(3\psi w_b t)\right].$$

4. Concluding remarks

In this paper, we implement ADM to solve the problem of the nonlinear vibrations of multiwalled carbon nanotubes. The advantage of the method is that it does not need a small parameter in the system, leading to wide application in nonlinear problems. Also, the convergence analysis of the proposed method is introduced. The numerical solutions have been compared with the results obtained via IHBM and excellent correlation has been obtained. The results clarify the significance dependency of the nonlinear free vibration of nanotubes to the surrounding elastic medium. The nonlinear vibration frequency of nanotubes rises rapidly with increasing the amplitude especially when the stiffness of the medium is relatively small. For larger stiffness (say $k > 10^9$ N/m^2), the nonlinear vibration tends to the linear regime. This method can be easily extended to the multiwalled CNTs with number of walls more than three. It is worthwhile to mention that ADM is straightforward and it is a promising and powerful technique for solving many nonlinear equations arising in mathematical physics.

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